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Numerical Tests of Upwind Scheme Performance for Entropy Condition

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I. Introduction

TO treat the flows with shock waves and contact discontinuities, an accurate and efficient upwind scheme used as a Riemann solver to resolve the discontinuities is essential. Such an upwind scheme is particularly important when it is incorporated into a high-order accuracy scheme such as an essentially nonoscillatory scheme to simulate turbulence or acoustic fields with discontinuities, where the nonphysical noise should be minimized and the number of mesh points would be limited due to the computing power.

To achieve efficiency and accuracy, efforts have been made to develop an upwind scheme only using scalar dissipation instead of using matrix dissipation such as that of Roe's[1] flux difference splitting (FDS) scheme. The modification of Van Leer's[2] flux vector splitting (FVS) scheme by Hänel et al.[3] in 1987 began a series of new developments. The advection upstream splitting method (AUSM) suggested by Liou and Steffen[4] in 1993 has achieved high accuracy while the computing work remains as low as that of the Van Leer[2] scheme. Zha and Bilgen[5] suggested a low diffusion FVS scheme in 1993, which may resolve crisp shock profiles, but the contact discontinuity would be smeared. The Zha–Bilgen scheme has been modified to accurately resolve contact discontinuities in Ref. 6. Jameson[7–9] suggested his convective upwind and split pressure (CUSP) schemes and limiters in 1993, which may capture crisp shock profiles, but not contact discontinuities. Liou's[10,11] AUSM$^+$ scheme further improves the accuracy of the AUSM scheme and is able to resolve the exact shock and contact discontinuities and to preserve the constant total enthalpy for steady-state flows. Using methodology similar to the AUSM scheme, Edwards[12,13] developed his low diffusion flux splitting scheme (LDFSS). The scheme shows the best performance in the numerical tests of this paper. The splitting formulations of Mach number and pressure from Van Leer's FVS scheme are employed in both AUSM and LDFSS schemes.

Although progress has been made to reduce the dissipation of upwind schemes and to improve their capability to capture discontinuities, relatively less attention has been paid to answer an important question: Does the scheme satisfy the entropy condition? For FDS schemes such as Roe's[1] scheme and Osher's[14] scheme, it is known that the Roe scheme does not satisfy the entropy condition and the Osher scheme does. For FVS schemes such as the Van Leer[2] scheme and the Steger–Warming[15] scheme, it is not known whether they satisfy the entropy condition.[16,17] The extensive and successful applications of the FVS schemes in practical computations lead to a general impression that a FVS scheme, in particular, the Van Leer scheme, is very robust, and violation of the entropy condition may not be an issue. However, this cannot serve as the answer to the question just raised. For those recently developed schemes, including AUSM family schemes,[10,11] LDFFS schemes,[12,13] CUSP schemes, modified Zha–Bilgen scheme,[6] etc., no definite answers are given regarding whether they satisfy the entropy condition.

The purpose of this Technical Note is to create a numerical test case that may answer the entropy condition question for some of the upwind schemes. If an upwind scheme generates an expansion shock, the scheme would be considered as violating the entropy condition. If an upwind scheme does not generate an expansion shock for the test case, it is not conclusive that the scheme satisfies the entropy condition. To explore the original properties of the upwind schemes, the work in this paper mainly focuses on the one-dimensional piecewise constant first-order discretization.

II. Numerical Procedure

The governing equations are the quasi-one-dimensional Euler equations in Cartesian coordinates:

$$\partial_t \boldsymbol{U} + \partial_x \boldsymbol{E} - \boldsymbol{H} = 0 \tag{1}$$

where

$$\boldsymbol{U} = S\boldsymbol{Q}, \qquad \boldsymbol{Q} = \begin{bmatrix} \rho \\ \rho u \\ e \end{bmatrix}, \qquad \boldsymbol{E} = S\boldsymbol{F}$$

$$\boldsymbol{F} = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ (e+p)u \end{bmatrix}, \qquad \boldsymbol{H} = \frac{\mathrm{d}S}{\mathrm{d}x}\begin{bmatrix} 0 \\ p \\ 0 \end{bmatrix}$$

In the preceding equations, ρ is the density, u is the velocity, p is the static pressure, e is the total energy, and S is the cross-sectional area of the one-dimensional duct. The following state equation is also employed:

$$p = (\gamma - 1)\left(e - \tfrac{1}{2}\rho u^2\right) \tag{2}$$

where γ is the ratio of specific heats and has the value 1.4.

The finite volume method with explicit Euler temporal integration is used to discretize the governing equations. It yields the following formulation at cell i:

$$\Delta \boldsymbol{Q}_i^{n+1} = \Delta t\left[-C\left(\boldsymbol{E}_{i+\frac{1}{2}} - \boldsymbol{E}_{i-\frac{1}{2}}\right) + (\boldsymbol{H}_i/S_i)\right]^n \tag{3}$$

where $C = 1/(\Delta x S_i)$ and n is the time level index.

The following upwind schemes are incorporated into this finite volume solver to evaluate the interface flux $\boldsymbol{F}_{i+1/2}$. They are the Van Leer[2] FVS scheme, the Roe[1] FDS scheme, the Van Leer–Hänel[3] scheme, the Steger–Warming[15] FVS scheme, the Liou[11] AUSM$^+$ scheme, the modified Zha–Bilgen scheme,[6] and the Edwards[12] LDFSS(2) scheme. In Ref. 6, the detailed formulations of each of the cited upwind schemes are listed for the purpose of reproducing the results.

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III. Results and Discussion

The test case is a simple quasi-one-dimensional transonic flow of a converging–diverging nozzle. The correct solution should be a smooth flow transitioning from subsonic to supersonic with no shock. However, for an upwind scheme that does not satisfy the entropy condition, an expansion shock may be produced. Because the purpose of this paper is not to cure the weakness of the upwind schemes tested, remedies[17,18] that modify the source terms thereof are not explored.

A. Geometry

The geometry is one of a series of two-dimensional converging–diverging nozzles designed and tested[19] at NASA Langley Research Center, namely, nozzle A2 as shown in Fig. 1. The geometry is formed by a plane upstream and downstream of the throat region with slope angles of θ and β, respectively. In the throat region, it has a circular-arc surface for transition. The geometry is symmetric about the central axis plane, and only the upper-half geometry is shown in Fig. 1. The formulations describing the geometry are

$$y = \tan(\theta)x + h_i, \qquad \text{for} \qquad 0 \le x \le L_1 \tag{4}$$

$$y = -\sqrt{r_c^2 - (x - x_c)^2} + y_c, \qquad \text{for} \qquad L_1 \le x \le L_2 \tag{5}$$

$$y = \tan(\beta)(x - x_t) + y_t, \qquad \text{for} \qquad L_2 \le x \le L_3 \tag{6}$$

where $\theta = -22.33$ deg, $\beta = 1.21$ deg, $L_1 = 4.74$ cm, $L_2 = 5.84$ cm, $L_3 = 11.56$ cm, $x_t = 5.84$ cm, $y_t = 1.37$ cm, $x_c = 5.78$ cm, $y_c = 4.11$ cm, $r_c = 2.74$ cm, and $h_i = 3.52$ cm.

In the computation, the geometry is normalized by the half-throat-height $h_t = 1.37$ cm.

B. Numerical Results

For the subsonic boundary conditions at the entrance, the velocity is extrapolated from the inner domain, and the other variables are determined by the total temperature (300 K) and total pressure (1013 kPa). For supersonic exit boundary conditions, all of the variables are extrapolated from inside of the nozzle. The analytical solution was used as the initial flowfield. The computation proceeded using a global time step in a time accurate fashion.

Figure 2 is the comparison of the analytical and computed Mach number distributions with 201 mesh points using the schemes[1–3,10,11] of Roe, Van Leer–Hänel, Liou AUSM, and Liou AUSM$^+$ before these computations diverged. The schemes, shown in Fig. 2, were either not stable in time or diverged for mesh refinement. The analytical solution is smooth throughout and reaches the sonic speed at the throat (the minimum area of the nozzle is located at $X/h_t = 4.22$). It is seen that all of these schemes generate expansion shocks at the nozzle throat location. The expansion shock of the Roe scheme had a large amplitude and was converged to machine zero with 201 mesh points and Courant–Friedrichs–Lewy number CFL = 0.95. Machine zero is defined as that the L2 norm residual is reduced at least by 12 orders of magnitude. When a refined mesh with 401 points was used, the amplitude of the expansion shock generated by the Roe scheme largely oscillated in time, and the calculation eventually diverged. The Van Leer–Hänel schemes[2,3] and Liou's[10,11] AUSM and AUSM$^+$ were not stable even for 201 mesh points with CFL number lower than 0.1. The amplitude of the expansion shock generated by Van Leer–Hänel and Liou's AUSM and

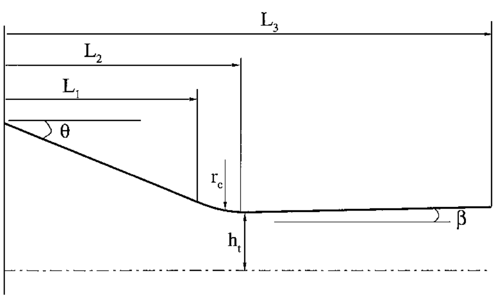

Fig. 1 NASA nozzle A2.

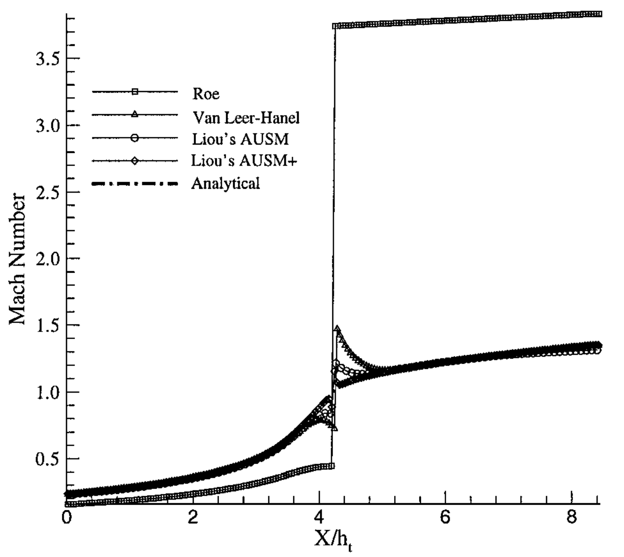

Fig. 2 Mach number distributions of the upwind schemes that are not stable: mesh points 201.

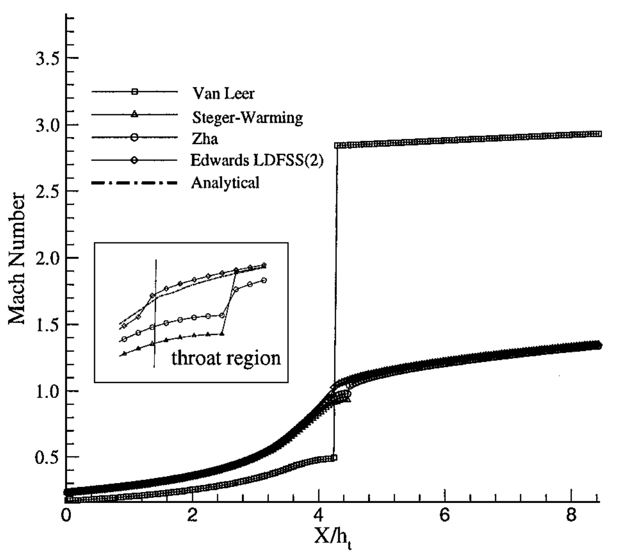

Fig. 3 Mach number distributions of the upwind schemes that are converged: mesh points 201.

AUSM$^+$ schemes grew in time, and the computation diverged soon after the results reached the state shown in Fig. 2.

Figure 3 is the Mach number distributions with 201 mesh points computed by the schemes of Van Leer[2] and of Steger–Warming,[15] the modified Zha–Bilgen scheme suggested in Ref. 6, and Edwards's[12] LDFSS(2) scheme. All of the results were converged to machine zero with CFL = 0.95. It is seen that the Van Leer scheme generates a strong expansion shock at the throat location. The derivatives of the modified[6] Zha–Bilgen and Steger–Warming[15] schemes are not continuous, and there are small jumps at the sonic points, as shown in the throat region. Edwards's[12,13] LDFSS(2) scheme agrees the best with the analytical results with the least jump. All of the schemes in Fig. 3 produce results closer to the analytical one when the refined mesh with 401 points is used.[6] The amplitude of the expansion shock generated by the Van Leer scheme is also decreased with the refined mesh, but remains very large.

Van Leer presented an analysis in Ref. 18 indicating that piecewise constant approximations may not be a good representation of the solution near a sonic point for the schemes that do not satisfy the entropy condition. Piecewise approximations prevent the true gradient being computed at the sonic bordering zone and shows up as a transonic expansion shock. For a second-order scheme, the expansion shock either disappears or its amplitude reduces to $\mathcal{O}[(\Delta x^3)]$ (for Burgers' equation). In Ref. 6, the results using the second-order

MUSCL-type interpolation shows that the expansion shocks indeed disappear and all of the schemes tested in this Note generate solutions that are smooth and virtually identical to the analytical solution. This implies that it may be generally safe to avoid an expansion shock as long as higher than first-order MUSCL-type upwind differencing is used.

IV. Conclusion

A numerical test case for the Euler equations is presented to examine the performance of several upwind schemes for satisfying the entropy condition. The first-order upwind schemes of Roe,[1] Van Leer,[2] Steger–Warming,[15] and Van Leer–Hänel et al.[3] and Liou's AUSM[4] and AUSM$^+$,[11] Edwards's LDFSS,[12,13] and a modified Zha–Bilgen[6] scheme are tested. The numerical test indicates that, similar to the Roe scheme, the Van Leer FVS scheme, the Van Leer–Hänel scheme, and AUSM-type schemes do not satisfy the entropy condition and allow expansion shocks at sonic point, when first-order accurate discretizations are used. The Roe scheme, AUSM, AUSM$^+$, and Van Leer–Hänel scheme are not stable due to the presence of the expansion shocks. The Van Leer scheme is stable but yields a strong expansion shock. The FVS scheme of Steger–Warming and the modified Zha–Bilgen scheme yield a small jump at the sonic point. The Edwards LDFSS(2) scheme gives the smoothest results at the sonic point with a very small glitch, when the coarse grid is used. A refined mesh decreases the nonsmoothness of the Steger–Warming scheme, the modified Zha–Bilgen scheme, and the Edwards LDFSS(2) scheme. When second-order MUSCL interpolations are used, all of the schemes obtain accurate results without any expansion shock and nonsmoothness at sonic point. This may imply that using higher than first-order spatial accuracy may generally avoid expansion shocks.

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Application of a Wavelet Cross-Correlation Technique to the Analysis of Mixing

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Introduction

THE analysis and control of mixing between two initially segregated streams is of great interest for aerospace, industrial, and environmental applications. A significant example is the mixing between two scalars seeded in two coaxial jets. Although the configuration is apparently simple from the geometrical point of view, the simultaneous presence of two shear layers leads to complex vorticity dynamics, characterized by nonlinear interactions between the structures forming from the instabilities of the two jets. As mixing is strictly connected with the vorticity dynamics, the mechanisms leading to mixing are complex.

The wavelet cross-correlation analysis[1] of the time signals of scalar concentrations can be an effective tool to study the mixing processes in this context. Indeed, this analysis permits the identification of the frequencies and of the time intervals at which the correlation between the fluctuations of the concentrations of the two scalars is high and, hence, mixing is significant. This technique is applied here to the time signals of scalar concentrations deriving from axisymmetric direct numerical simulation of a coaxial jet flow. Because we know that three-dimensional mechanisms play an important role even in the near field of coaxial jets, axisymmetric simulations cannot provide a detailed realistic description of a coaxial jet flow. However, they are well suited to characterize the capabilities of different processing techniques, which can be eventually applied also to three-dimensional simulations to analyze the mixing mechanisms. Moreover, numerical simulation provides simultaneously the time evolution of scalar concentrations and of the vorticity field, so that the mixing processes can be connected to the different events in the vorticity dynamics, such as roll up, passage, and pairing of vortices. Finally, as shown in Ref. 1, the wavelet cross-correlation analysis, if applied to the velocity signals, gives a clear description of the average and instantaneous contributions to the Reynolds stresses from the different events in the vorticity dynamics, which permits us to investigate if the events responsible for mixing are the same giving significant contribution to the Reynolds stress. In particular, although in the simplified context of axisymmetric flows, indications are given on the contribution of the roll up, passage, and pairing of vortical structures to Reynolds stress and to mixing between the two streams.

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